# Geometry of entanglement in the Bloch sphere

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Entanglement is an important concept in quantum information, quantum communication, and quantum computing. We provide a geometrical analysis of entanglement and separability for *all* the rank 2 quantum mixed states: complete analysis for the bipartite states and partial analysis for the multipartite states. For each rank 2 mixed state, we define its unique Bloch sphere, that is spanned by the eigenstates of its density matrix. We characterize those Bloch spheres into exactly five classes of entanglement and separability, give examples for each class, and prove that those are the only classes.

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## I. INTRODUCTION

Entanglement is a very important property of quantum states, relevant to the foundations of quantum mechanics (e.g., the Einstein-Podolsky-Rosen paradox and Bell's inequality), as well as to quantum information, quantum communication (including quantum teleportation and quantum cryptography), quantum computers and simulators, and quantum many-body systems.

The relations among entanglement, partial transpose, and nonclassical correlations between the subsystems, are well understood for pure quantum bipartite states. However, for mixed quantum states there are still many open questions. Even bipartite mixed states of rank 2 (namely, states that can be written as $\rho = p|\varphi\rangle\langle\varphi| + (1-p)|\psi\rangle\langle\psi|$, where $0 < p < 1$, and $|\varphi\rangle,|\psi\rangle$ are bipartite orthonormal states and are the eigenstates of $\rho$), that are discussed in this paper, are not well understood. Studying such states is thus a major challenge in the field of mixed-state quantum entanglement.

It is known that if a mixed state does not have a positive partial transpose then it is entangled and presents a nonlocal behavior [1]. However, one can find separable states presenting a nonlocal behavior (e.g., Ref. [2]), and one can find entangled states that have a positive partial transpose [3]; those states are bound entangled, namely, their entanglement cannot be distilled [4]. It was later proved that bound entangled states cannot have rank 3 or less [5,6]. Therefore, checking whether a *specific* rank 2 state is entangled is trivial: It is entangled if and only if it does not have a positive partial transpose. However, in this paper we discuss the problem of classifying each rank 2 state by checking which states in its *Bloch-sphere neighborhood* (namely, in its corresponding Bloch sphere) are entangled.

Entanglement distillation (for pure states) [7] and entanglement purification (for mixed states) [8] are processes of distilling Bell states (or other maximally entangled states) from some copies of an initial state. An efficient protocol is known for pure states but not for mixed states. This provides another motivation for studying and finding ways to fully characterize the simplest nonpure bipartite states (the rank 2 bipartite mixed states).

The notion of the Bloch sphere, also known as the Poincaré sphere, is a very useful geometrical interpretation of a single qubit. It can be extended to any 2-dimensional (complex) *subspace* of a full Hilbert space—for example, the subspace spanned by the eigenstates of any given rank 2 mixed state.

We define here the Bloch-sphere entanglement of a quantum rank 2 bipartite state. This (informally) means that we define the sets of separable states and of entangled states inside the unique Bloch sphere associated with this quantum state. We provide some examples, and we prove that the five classes we present exhaust all the possibilities of Bloch-sphere entanglement. We briefly discuss going beyond bipartite states, and we briefly present an interesting exception (from the above classification) for the case of just two qubits.

We primarily use the Peres-Horodecki criterion [1,3]: *If* for a state $\rho$ of the system $AB$, the operator $\rho^{T_B}$ is not positive semidefinite (where $\rho^{T_B}$ is the partial transpose of $\rho$ with respect to the the subsystem $B$), *then* $\rho$ is entangled.

It was shown in Ref. [3] that for systems of dimensions $2\otimes 2$, $2\otimes 3$, or $3\otimes 2$, $\rho$ is entangled *if and only if* $\rho^{T_B}$ is not positive semidefinite. Yet in higher dimensions there are entangled states (that are bound entangled states) that have a positive partial transpose [3,9].

In Sec. II we present a weaker entanglement criterion that we will use for proving our claims, and in Sec. III we introduce several important properties of Bloch spheres to be used in our proofs. In Sec. IV we present a classification of all rank 2 states into five classes, and in Sec. V we prove that no other classes exist. In Sec. VI we prove that one of the classes does not exist in a specific case (the two-qubit case). In Sec. VII we generalize some of our results to multipartite entanglement. In Sec. VIII we describe previous works in this area, and in Sec. IX we conclude.

## II. A WEAKER ENTANGLEMENT CRITERION

We will use this weaker entanglement criterion to prove our claims:

*Lemma 1.* Let $\rho^{AB}$ be a state of a bipartite system. **If** there are states $|\varphi_A\rangle$, $|\varphi_B\rangle$, $|\psi_A\rangle$, and $|\psi_B\rangle$ such that $\langle\varphi_A\varphi_B|\rho^{AB}|\varphi_A\varphi_B\rangle = 0$ and $\langle\varphi_A\psi_B|\rho^{AB}|\psi_A\varphi_B\rangle \neq 0$, **then** $\rho^{AB}$ is entangled.

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*Proof.* Let $\rho = \rho^{AB}$, $|\varphi\rangle = |\varphi_A \varphi_B^*\rangle$, and $|\psi\rangle = |\psi_A \psi_B^*\rangle$, where $|\varphi_B^*\rangle$ and $|\psi_B^*\rangle$ are obtained from $|\varphi_B\rangle$ and $|\psi_B\rangle$ by replacing their amplitudes in the standard (computational) basis by their complex conjugates.

We first need a property of $\rho^{T_B}$. By definition, the partial transpose of $C_{ijkl} = |i\rangle\langle j| \otimes |k\rangle\langle l|$ is $C_{ijkl}^{T_B} = |i\rangle\langle j| \otimes |l\rangle\langle k|$, and the partial transpose $\rho^{T_B}$ of $\rho$ is obtained by a linear extension. Therefore, for $C_{ijkl}$ it holds that

$$\begin{aligned}\langle \varphi_A \varphi_B^* | C_{ijkl}^{T_B} | \psi_A \psi_B^* \rangle &= \langle \varphi_A | i \rangle \langle j | \psi_A \rangle \langle \varphi_B^* | l \rangle \langle k | \psi_B^* \rangle \\ &= \langle \varphi_A | i \rangle \langle j | \psi_A \rangle \langle \psi_B | k \rangle \langle l | \varphi_B \rangle \\ &= \langle \varphi_A \psi_B | C_{ijkl} | \psi_A \varphi_B \rangle,\end{aligned}$$

and by linearity,

$$\langle \varphi_A \varphi_B^* |\, \rho^{T_B} \,| \psi_A \psi_B^* \rangle = \langle \varphi_A \psi_B |\, \rho \,| \psi_A \varphi_B \rangle.$$

If the condition of the lemma is satisfied, then $\langle \varphi_A \varphi_B^* | \rho^{T_B} | \varphi_A \varphi_B^* \rangle = \langle \varphi_A \varphi_B | \rho | \varphi_A \varphi_B \rangle = 0$ and $\langle \varphi_A \varphi_B^* | \rho^{T_B} | \psi_A \psi_B^* \rangle = \langle \varphi_A \psi_B | \rho | \psi_A \varphi_B \rangle \neq 0$. From Lemma 2 it follows that $\rho^{T_B}$ is not positive semidefinite, and thus that $\rho$ is entangled. ■

We declare this lemma to be a weaker criterion because it proves entanglement only for a subclass of all the states satisfying the Peres-Horodecki criterion.

*Lemma 2.* If a Hermitian operator $A$ is positive semidefinite and $\langle \varphi | A | \varphi \rangle = 0$, then $\langle \varphi | A | \psi \rangle = 0$ for all $|\psi\rangle$.

*Proof.* Let $A = \sum_i \lambda_i |i\rangle\langle i|$ with $\lambda_i \geqslant 0$; $\langle \varphi | A | \varphi \rangle = \sum_i \lambda_i |\langle \varphi | i \rangle|^2 = 0$ and thus $\langle \varphi | i \rangle = 0$ if $\lambda_i \neq 0$. It follows that $\langle \varphi | A | \psi \rangle = \sum_i \lambda_i \langle \varphi | i \rangle \langle i | \psi \rangle = 0$ for all $|\psi\rangle$. ■

Lemma 2 was presented by us (M.B. and T.M.) in a conference [10].

## III. PROPERTIES OF SUBSPACES AND BLOCH SPHERES

In the next sections, we also use the following results that were mentioned in Ref. [11]:

*Lemma 3.* Let $\mathcal{H}'$ be a subspace of a Hilbert space $\mathcal{H}$. Let $\rho \in \mathcal{L}(\mathcal{H}')$ (i.e., $\rho$ can be decomposed as a mixture of pure states from $\mathcal{H}'$). If $\rho = \sum_j q_j |\varphi_j\rangle\langle\varphi_j|$ is a decomposition of $\rho$ with $|\varphi_j\rangle \in \mathcal{H}$ and $q_j > 0$, then $|\varphi_j\rangle \in \mathcal{H}'$ for all $j$.

*Proof.* Let $\{|\psi_i\rangle\}_{i \in I'}$ be an orthonormal basis of $\mathcal{H}'$, and let us extend it to an orthonormal basis $\{|\psi_i\rangle\}_{i \in I}$ of $\mathcal{H}$ ($I' \subseteq I$). Let $|\varphi_j\rangle = \sum_{i \in I} a_{ji} |\psi_i\rangle$, with $a_{ji} = \langle \psi_i | \varphi_j \rangle$. Then for all $i \in I \setminus I'$,

$$0 = \langle \psi_i | \rho | \psi_i \rangle = \sum_j q_j \langle \psi_i | \varphi_j \rangle \langle \varphi_j | \psi_i \rangle = \sum_j q_j |a_{ji}|^2,$$

implying that $a_{ji} = 0$ for all $i \in I \setminus I'$, and thus $|\varphi_j\rangle = \sum_{i \in I'} a_{ji} |\psi_i\rangle \in \mathcal{H}'$. ■

*Corollary 4.* If a rank 2 mixed state $\rho$ is inside a specific Bloch sphere, then all the pure states in all of its decompositions lie on the same Bloch sphere.

By using Corollary 4, we get the following:

*Corollary 5.* If $\rho$ is a rank 2 mixed state, then it lies inside a unique Bloch sphere (the uniqueness is up to a possible rotation of the sphere).

*Corollary 6.* If a rank 2 mixed state $\rho$ is separable, then there exist at least two different pure separable states on its unique Bloch sphere.

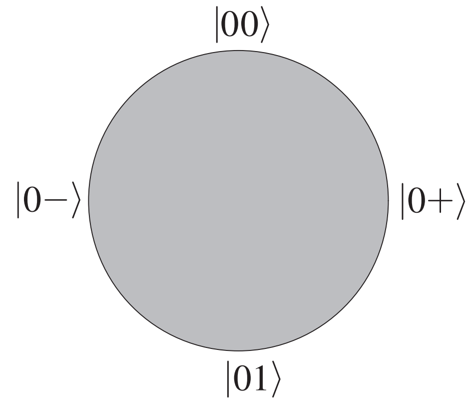


FIG. 1. Bloch sphere of the example for class 1: All the states on and inside this Bloch sphere are separable.

## IV. CLASSIFICATION OF BLOCH-SPHERE ENTANGLEMENT

In the rest of this paper we use Lemma 1 (a weaker entanglement criterion), Lemma 2 (a positive semidefinite operators condition), Corollary 5 (the unique-Bloch-sphere corollary), and Corollary 6 (a separable states condition) in order to provide a classification of Bloch-sphere entanglement. This is based on the following understanding: If $\rho$ is a bipartite rank 2 mixed state that is a mixture of pure states in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, then according to Corollary 5, it lies inside a unique Bloch sphere (the uniqueness is up to a possible rotation); and this Bloch sphere corresponds to a 2-dimensional subspace of $\mathcal{H}_A \otimes \mathcal{H}_B$.

We present five different classes of 2-dimensional subspaces of a bipartite system that are distinguished by their Bloch-sphere entanglement. (It is sufficient to consider only examples for which $\mathcal{H}_A$ is 2 dimensional ($\mathcal{H}_2$) and $\mathcal{H}_B$ is either 2 dimensional ($\mathcal{H}_2$) or 3 dimensional ($\mathcal{H}_3$)).

(1) No entanglement at all

Example in $\mathcal{H}_2 \otimes \mathcal{H}_2$: Span$\{|00\rangle, |01\rangle\}$ (Fig. 1)

(2) Entanglement everywhere on and inside the sphere except a line (of separable states) connecting two *orthogonal* pure states on the sphere (e.g., the poles)

Example in $\mathcal{H}_2 \otimes \mathcal{H}_2$: Span$\{|00\rangle, |11\rangle\}$ (Fig. 2)

(3) Entanglement everywhere on and inside the sphere except a line (of separable states) connecting two *nonorthogonal* pure states on the sphere

Example in $\mathcal{H}_2 \otimes \mathcal{H}_2$: Span$\{|00\rangle, |{+}{+}\rangle\}$ (Fig. 3)

(4) Entanglement everywhere on and inside the sphere except a single separable point on the sphere

Example in $\mathcal{H}_2 \otimes \mathcal{H}_2$: Span$\{|00\rangle, \alpha|01\rangle + \beta|10\rangle\}$ with $\alpha\beta \neq 0$ (Fig. 4 and Proposition 7)

(5) Entanglement everywhere ("completely entangled subspace")

Example in $\mathcal{H}_2 \otimes \mathcal{H}_3$: Span$\{[|00\rangle + |11\rangle]/\sqrt{2}, [|02\rangle + |10\rangle]/\sqrt{2}\}$ (Fig. 5 and Proposition 8)

Does not exist in $\mathcal{H}_2 \otimes \mathcal{H}_2$ (proof is given in Sec. VI, as Proposition 10)

Very similar examples can be found in all the bipartite Hilbert spaces (if the dimensions of both subsystems are at least 2), except the example of class 5, that does not exist in $\mathcal{H}_2 \otimes \mathcal{H}_2$.

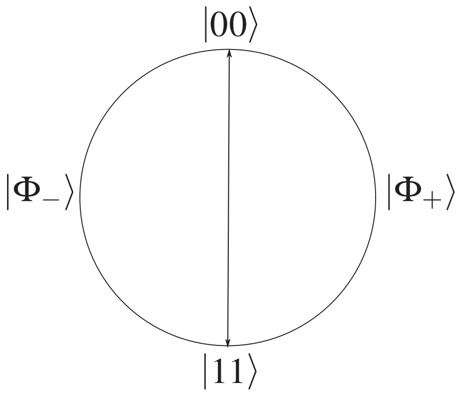


FIG. 2. Bloch sphere of the example for class 2: All the states along the line connecting $|00\rangle$ and $|11\rangle$ are separable; all the other states on and inside this Bloch sphere are entangled. Any two orthogonal product states can replace $|00\rangle$ and $|11\rangle$.

The analysis of classes 1–3 (see Figs. 1–3) is very simple and follows directly from the proof of the general Theorem 9. Generally speaking, if two pure separable states exist on the Bloch sphere, then it belongs to one of those classes.

We now analyze the example for class 4 (see Fig. 4), a class that we found, that was also found independently by authors of Ref. [12].

*Proposition 7.* Let $|\psi_0\rangle = |00\rangle$ and $|\psi_1\rangle = \alpha|01\rangle + \beta|10\rangle$ with $\alpha\beta \neq 0$, $|\alpha|^2 + |\beta|^2 = 1$. The state $\rho = a_{00}|\psi_0\rangle\langle\psi_0| + a_{01}|\psi_0\rangle\langle\psi_1| + a_{10}|\psi_1\rangle\langle\psi_0| + a_{11}|\psi_1\rangle\langle\psi_1|$ is separable if and only if $a_{01} = a_{10} = a_{11} = 0$.

*Proof.* $\langle 11|\rho|11\rangle = 0$ and $\langle 10|\rho|01\rangle = a_{11}\langle 10|\psi_1\rangle\langle\psi_1|01\rangle = a_{11}\beta\alpha^*$; thus, by Lemma 1 (the weaker entanglement criterion), $\rho$ is entangled if $a_{11} \neq 0$. If $a_{11} = 0$, $\langle\psi_1|\rho|\psi_1\rangle = 0$ and $\langle\psi_1|\rho|\psi_0\rangle = a_{10}$; therefore, by Lemma 2, $a_{10} = 0$, which implies that $a_{01} = a_{10}^* = 0$. ■

Finally, for the example of class 5 (see Fig. 5) see Proposition 8:

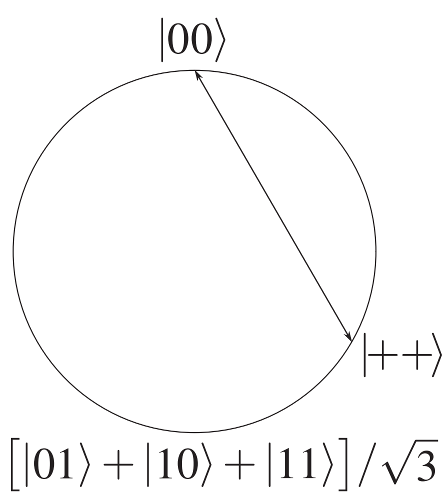


FIG. 3. Bloch sphere of the example for class 3: All the states along the line connecting $|00\rangle$ and $|{+}{+}\rangle$ are separable; all the other states on and inside this Bloch sphere are entangled. Any two nonorthogonal linearly independent product states can replace $|00\rangle$ and $|{+}{+}\rangle$.

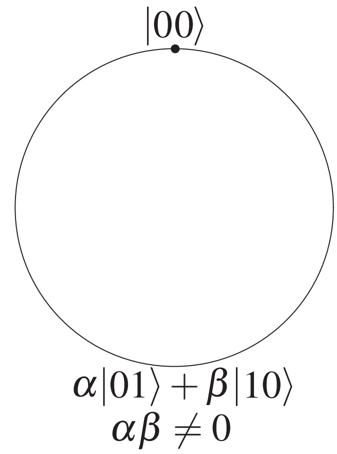


FIG. 4. Bloch sphere of the example for class 4: Only the state $|00\rangle$ is separable; all the other states on and inside this Bloch sphere are entangled.

*Proposition 8.* Let $|\psi_0\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ and $|\psi_1\rangle = (|02\rangle + |10\rangle)/\sqrt{2}$. The state $\rho = a_{00}|\psi_0\rangle\langle\psi_0| + a_{01}|\psi_0\rangle\langle\psi_1| + a_{10}|\psi_1\rangle\langle\psi_0| + a_{11}|\psi_1\rangle\langle\psi_1|$ is always entangled.

*Proof.* By using Corollary 6 (the separable states condition), it is sufficient to prove that all the pure states $|\psi\rangle = \alpha|\psi_0\rangle + \beta|\psi_1\rangle$ are entangled.

Let us look at the state

$$\begin{aligned}|\psi\rangle &= \alpha|\psi_0\rangle + \beta|\psi_1\rangle\\ &= \frac{\alpha}{\sqrt{2}}|00\rangle + \frac{\alpha}{\sqrt{2}}|11\rangle + \frac{\beta}{\sqrt{2}}|02\rangle + \frac{\beta}{\sqrt{2}}|10\rangle\\ &\triangleq \sum_{i,j}\epsilon_{ij}|i\rangle|j\rangle.\end{aligned}$$

For this state to be separable, there must exist $a_0, a_1$ and $b_0, b_1, b_2$ such that $\epsilon_{ij} = a_i b_j$ for all $i, j$; hence, the equations $\epsilon_{01} = a_0 b_1 = 0$ and $\epsilon_{12} = a_1 b_2 = 0$ must hold. By a simple calculation it follows that necessarily $\alpha = \beta = 0$, which is impossible. We conclude that there are no separable pure states on the Bloch sphere, and thus there are no separable mixed states. ■

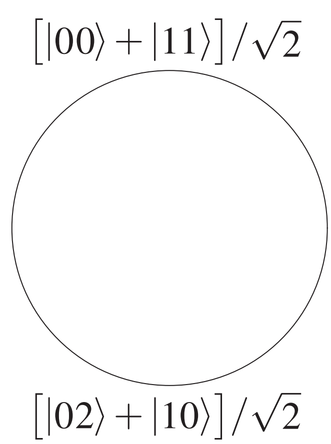


FIG. 5. Bloch sphere of the example for class 5: All the states on and inside this Bloch sphere are entangled.

Our classification suggests natural ways to measure entanglement inside the Bloch sphere: For example, entanglement may be measured by the Euclidean distance to the closest separable state (e.g., given the Bloch sphere Span$\{|00\rangle,|11\rangle\}$, the closest separable state to the pure state $\alpha|00\rangle+\beta|11\rangle$ is the state $|\alpha|^2|00\rangle\langle 00|+|\beta|^2|11\rangle\langle 11|$). We note that this entanglement measure, unlike the measures analyzed by Refs. [11,13], vanishes only for separable states. Analyzing the properties of such measures is beyond the scope of this paper.

## V. A PROOF THAT THERE ARE EXACTLY FIVE CLASSES OF BLOCH-SPHERE ENTANGLEMENT

Our main goal is to provide a full analysis of the general bipartite case. We prove that the classes we found are the only classes that exist in the bipartite case, for all the rank 2 bipartite states (namely, for all the corresponding 2-dimensional Hilbert spaces):

*Theorem 9.* Let $\mathcal{H}$ be a 2-dimensional subspace of $\mathcal{H}_A\otimes\mathcal{H}_B$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are two Hilbert spaces. Then $\mathcal{H}$ belongs to one of the following classes:

Class 1: The Bloch ball of $\mathcal{H}$ is completely separable.

Class 2+3: The Bloch ball of $\mathcal{H}$ has one line of separable states, and all the other states are entangled.

Class 4: The Bloch ball of $\mathcal{H}$ has one separable point (pure state), and all the other states are entangled.

Class 5: The Bloch ball of $\mathcal{H}$ is completely entangled.

(We note that class 2 and class 3 are discussed together, because in both of them the Bloch ball has just one line of separable states.)

*Proof.* First, assume that there is no separable *mixed* state inside the Bloch ball. This means that there is at most one pure separable state on the Bloch sphere (because if two pure states are separable, then the line connecting them inside the Bloch ball is separable, too). This matches classes 4 and 5.

Now assume that there is a separable *mixed* state $\rho$ inside the Bloch ball. According to Corollary 6 (the separable states condition), this means that there are at least two different pure separable states on the Bloch sphere. We denote them by $|\psi\rangle=|\psi_A\rangle\otimes|\psi_B\rangle$ and $|\varphi\rangle=|\varphi_A\rangle\otimes|\varphi_B\rangle$.

We note that $|\psi\rangle\ncong|\varphi\rangle$ (defining the symbol $\cong$ to be equality as normalized states, possibly with different global phases; thus, the symbol $\ncong$ means that the two normalized states are really different, as opposed to states that are equal up to a global phase), which means that $|\psi\rangle$ and $|\varphi\rangle$ are linearly independent. Therefore, the Bloch sphere represents the 2-dimensional subspace Span$\{|\psi\rangle,|\varphi\rangle\}$, which means that all the mixed states inside the Bloch ball are of the form

$$\rho=a_{00}|\psi\rangle\langle\psi|+a_{01}|\psi\rangle\langle\varphi|+a_{10}|\varphi\rangle\langle\psi|+a_{11}|\varphi\rangle\langle\varphi|. \quad (1)$$

If $|\psi_A\rangle\cong|\varphi_A\rangle$ or $|\psi_B\rangle\cong|\varphi_B\rangle$, then obviously all the states on and inside the Bloch sphere are separable, which matches class 1.

If $|\psi_A\rangle\ncong|\varphi_A\rangle$ and $|\psi_B\rangle\ncong|\varphi_B\rangle$, then we prove that only the line connecting $|\psi\rangle$ and $|\varphi\rangle$ inside the Bloch ball is separable and that all the other pure and mixed states in the Bloch ball are entangled. This will match classes 2 and 3, and will conclude our proof.

We look at all the mixed states of the form (1). If $a_{01}=a_{10}=0$, then we obviously get a separable state:

$$\rho=a_{00}|\psi_A\rangle\langle\psi_A|\otimes|\psi_B\rangle\langle\psi_B|+a_{11}|\varphi_A\rangle\langle\varphi_A|\otimes|\varphi_B\rangle\langle\varphi_B|.$$

If $a_{10}\neq 0$, then let $|\overline{\varphi_A}\rangle\in\mathcal{H}_A$ satisfy $\langle\varphi_A|\overline{\varphi_A}\rangle=0$ and $\langle\psi_A|\overline{\varphi_A}\rangle\neq 0$ ($|\overline{\varphi_A}\rangle$ always exists, because $|\psi_A\rangle\ncong|\varphi_A\rangle$). Similarly, let $|\overline{\psi_A}\rangle\in\mathcal{H}_A$ and $|\overline{\varphi_B}\rangle,|\overline{\psi_B}\rangle\in\mathcal{H}_B$ satisfy similar properties (because $|\psi_B\rangle\ncong|\varphi_B\rangle$). Then

$$\langle\overline{\psi_A}\ \overline{\varphi_B}|\rho|\overline{\psi_A}\ \overline{\varphi_B}\rangle=0$$

and

$$\begin{aligned}\langle\overline{\psi_A}\ \overline{\psi_B}|\rho|\overline{\varphi_A}\ \overline{\varphi_B}\rangle&=a_{10}\langle\overline{\psi_A}\ \overline{\psi_B}|\varphi_A\varphi_B\rangle\langle\psi_A\psi_B|\overline{\varphi_A}\ \overline{\varphi_B}\rangle\\&=a_{10}\langle\overline{\psi_A}|\varphi_A\rangle\langle\overline{\psi_B}|\varphi_B\rangle\langle\psi_A|\overline{\varphi_A}\rangle\langle\psi_B|\overline{\varphi_B}\rangle\\&\neq 0.\end{aligned}$$

Therefore, by Lemma 1 (the weaker entanglement criterion), if $a_{10}\neq 0$ (or $a_{01}\neq 0$; this is equivalent, because $a_{01}=a_{10}^*$), then $\rho$ is entangled.

We conclude that only the line between $|\psi\rangle$ and $|\varphi\rangle$ (i.e., the line of states satisfying $a_{01}=a_{10}=0$) is separable and that the other states (i.e., the states satisfying $a_{10}\neq 0$ or $a_{01}\neq 0$) are entangled, which matches classes 2 and 3. This concludes our proof. ■

## VI. A PROOF THAT CLASS 5 DOES NOT EXIST IN THE TWO-QUBIT CASE

We have seen that for almost all the bipartite Hilbert spaces, five classes appear. We now show that for the Hilbert space $\mathcal{H}_2\otimes\mathcal{H}_2$, only four classes exist (classes 1–4):

*Proposition 10.* No 2-dimensional subspace of $\mathcal{H}_2\otimes\mathcal{H}_2$ is completely entangled.

*Proof.* This proof follows the methods of Ref. [11]. We remember that for a two-qubit state $|\psi\rangle=\sum_{i,j}a_{ij}|i\rangle|j\rangle$, the concurrence $C$ is defined as follows [14,15]:

$$C(\psi)=2|a_{00}a_{11}-a_{01}a_{10}|. \quad (2)$$

In particular, $C(\psi)=0$ if and only if $|\psi\rangle$ is separable. (This is not necessarily true for other entanglement measures.)

Let $\mathcal{H}\triangleq$ Span$\{|\psi_0\rangle,|\psi_1\rangle\}$ be a 2-dimensional subspace of $\mathcal{H}_2\otimes\mathcal{H}_2$. We may assume that $C(\psi_1)\neq 0$ (otherwise, $|\psi_1\rangle$ is separable, hence $\mathcal{H}$ cannot be completely entangled). Therefore, the set of separable (non-normalized) pure states in $\mathcal{H}$ is the set of states $|\psi_0\rangle+z|\psi_1\rangle$ satisfying the following equation:

$$C(|\psi_0\rangle+z|\psi_1\rangle)=0.$$

This is a quadratic equation in the complex variable $z$ (because we may ignore the absolute value). The absolute value of the coefficient of $z^2$ is $C(\psi_1)\neq 0$. Therefore, there are two solutions $\xi_1,\xi_2$ (possibly equal) to this equation, and thus the non-normalized state $|\psi_0\rangle+\xi_1|\psi_1\rangle$ (whose normalization is in $\mathcal{H}$) must be separable. Therefore, there is a separable state in $\mathcal{H}$, and $\mathcal{H}$ cannot be completely entangled. ■

## VII. EXAMPLES AND ANALYSIS OF MULTIPARTITE ENTANGLEMENT

For multipartite states, there are several different definitions of separability and entanglement: An $m$-partite mixed state is "fully separable" if it is a mixture of pure states that are products of $m$ pure states; and it is "separable with respect to a bipartite partition $\mathcal{P}$" (with $\mathcal{P}$ partitioning the $m$ subsystems into two disjoint sets) if the bipartite state corresponding to the partition $\mathcal{P}$ is separable [16]. For example, the state $|0\rangle_A|\Phi_+\rangle_{BC} \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ is separable with respect to the partition $\{\{1\},\{2,3\}\}$, but is entangled with respect to both partitions $\{\{1,2\},\{3\}\}$ and $\{\{1,3\},\{2\}\}$. Note that even if a state is separable with respect to all the bipartite partitions, it may still be entangled (i.e., not fully separable) [9].

To illustrate the many existing possibilities for Bloch spheres in the multipartite case, we look at two examples:

(1) $\text{Span}\{|000\rangle,|111\rangle\}$: the line connecting between the north pole ($|000\rangle$) and the south pole ($|111\rangle$) is fully separable; all the other points are entangled with respect to *any* bipartite partition.

(2) $\text{Span}\{|000\rangle,|011\rangle\}$: the line connecting between the north pole ($|000\rangle$) and the south pole ($|011\rangle$) is fully separable; all the other points are separable with respect to the bipartite partition $\{\{1\},\{2,3\}\}$, but are entangled with respect to the partitions $\{\{1,2\},\{3\}\}$ and $\{\{1,3\},\{2\}\}$.

The proofs of separability above are direct from the definitions; and the proofs of entanglement are implied by our analysis in the proof of Theorem 9.

Moreover, our Theorem 9 is true also for the set of *fully separable* states in the multipartite case:

*Theorem 11.* Let $\mathcal{H}$ be a 2-dimensional subspace of $\mathcal{H}_{A_1} \otimes \cdots \otimes \mathcal{H}_{A_m}$, where $\mathcal{H}_{A_1},\ldots,\mathcal{H}_{A_m}$ are Hilbert spaces. Then $\mathcal{H}$ belongs to one of the following classes:

Class 1: All the states inside the Bloch ball of $\mathcal{H}$ are fully separable.

Class 2+3: The Bloch ball of $\mathcal{H}$ has one line of fully separable states, and all the other states are not fully separable.

Class 4: The Bloch ball of $\mathcal{H}$ has one fully separable point (pure state), and all the other states are not fully separable.

Class 5: All the states inside the Bloch ball of $\mathcal{H}$ are not fully separable.

*Proof.* First, assume that there is no fully separable *mixed* state inside the Bloch ball. This means that there is at most one pure fully separable state on the Bloch sphere (because if two pure states are fully separable, then the line connecting them inside the Bloch ball is fully separable, too). This matches classes 4 and 5.

Now assume that there is a fully separable *mixed* state $\rho$ inside the Bloch ball. According to Corollary 6 (the separable states condition), this means that there are at least two different fully separable pure states on the Bloch sphere. We denote them by $|\psi\rangle = |\psi_{A_1}\rangle \otimes \cdots \otimes |\psi_{A_m}\rangle$ and $|\varphi\rangle = |\varphi_{A_1}\rangle \otimes \cdots \otimes |\varphi_{A_m}\rangle$.

We note that $|\psi\rangle \ncong |\varphi\rangle$ (defining the symbol $\cong$ as we did in the proof of Theorem 9 above; thus, the symbol $\ncong$ means that the two normalized states are really different, as opposed to states that are equal up to a global phase), which means that $|\psi\rangle$ and $|\varphi\rangle$ are linearly independent. Therefore, the Bloch sphere represents the 2-dimensional subspace $\text{Span}\{|\psi\rangle,|\varphi\rangle\}$, which means that all the mixed states inside the Bloch ball are of the form

$$\rho = a_{00}|\psi\rangle\langle\psi| + a_{01}|\psi\rangle\langle\varphi| + a_{10}|\varphi\rangle\langle\psi| + a_{11}|\varphi\rangle\langle\varphi|. \quad (3)$$

If $|\psi_{A_i}\rangle \cong |\varphi_{A_i}\rangle$ for all $i$ except one value of $i$, then obviously all the states on and inside the Bloch sphere are fully separable, which matches class 1.

If $|\psi_{A_{i_1}}\rangle \ncong |\varphi_{A_{i_1}}\rangle$ and $|\psi_{A_{i_2}}\rangle \ncong |\varphi_{A_{i_2}}\rangle$ for $i_1 < i_2$, then we prove that for the bipartite partition $\{I_1,I_2\}$ with $I_1 = \{1,\ldots,i_1\}$ and $I_2 = \{i_1+1,\ldots,m\}$ (satisfying $I_1 \cup I_2 = \{1,\ldots,m\}$, $I_1 \cap I_2 = \emptyset$, $i_1 \in I_1$, and $i_2 \in I_2$), it holds that only the line connecting $|\psi\rangle$ and $|\varphi\rangle$ inside the Bloch ball is fully separable, and that all the other pure and mixed states in the Bloch ball are entangled with respect to the partition $\{I_1,I_2\}$. This will match classes 2 and 3, and will conclude our proof.

To prove that the line is fully separable, we notice that any convex combination of fully separable states is fully separable, and therefore the line connecting $|\psi\rangle$ and $|\varphi\rangle$ inside the Bloch ball is fully separable.

To prove that all the other states are entangled with respect to the partition $\{I_1,I_2\}$, we denote $|\psi^{I_1}\rangle = |\psi_{A_1}\rangle \otimes \cdots \otimes |\psi_{A_{i_1}}\rangle$ and $|\psi^{I_2}\rangle = |\psi_{A_{i_1+1}}\rangle \otimes \cdots \otimes |\psi_{A_m}\rangle$; and similarly, we define $|\varphi^{I_1}\rangle$ and $|\varphi^{I_2}\rangle$. Then, because $i_1 < i_2$, and because $|\psi_{A_{i_1}}\rangle \ncong |\varphi_{A_{i_1}}\rangle$ and $|\psi_{A_{i_2}}\rangle \ncong |\varphi_{A_{i_2}}\rangle$, it must hold that $|\psi^{I_1}\rangle \ncong |\varphi^{I_1}\rangle$ and $|\psi^{I_2}\rangle \ncong |\varphi^{I_2}\rangle$. It also holds that $|\psi\rangle = |\psi^{I_1}\rangle \otimes |\psi^{I_2}\rangle$ and $|\varphi\rangle = |\varphi^{I_1}\rangle \otimes |\varphi^{I_2}\rangle$; therefore, according to the proof of the original Theorem 9, it holds that all the states outside of the line connecting $|\psi\rangle$ and $|\varphi\rangle$ in the Bloch ball (i.e., all the states satisfying $a_{01} \neq 0$ or $a_{10} \neq 0$) are entangled with respect to the partition $\{I_1,I_2\}$. Together with the proof that all the states on that line (i.e., all the states satisfying $a_{01} = a_{10} = 0$) are fully separable, this matches classes 2 and 3, and concludes our proof. ■

Extensions of Theorem 9 to other cases of multipartite entanglement are beyond the scope of this paper.

## VIII. PREVIOUS WORKS

The existence of completely entangled subspaces has been discussed in many papers before. In particular, this notion was used in Ref. [9] to prove the existence of a huge class of bound entangled states.

Analysis of entangled states in a Hilbert subspace, using *specific* entanglement measures (e.g., the concurrence and the 3-tangle) and Bloch spheres, was done by Refs. [13] and [11]. However, the entanglement measures they choose usually vanish not only for all the separable states, but also for some of the entangled states [11]. Much more recently, Refs. [12] and [17] investigated interesting classes in the *same* research direction. In contrast, our paper analyzes the separability and the entanglement in the Bloch sphere for any rank 2 bipartite state; and, instead of using a specific entanglement measure that *cannot show the entanglement* of some of the entangled states, we fully characterize the set of separable states on and inside the state's Bloch sphere.

## IX. CONCLUSION

We have found a complete classification of the possible sets of separable states in all the 2-dimensional subspaces of bipartite Hilbert spaces. Our result is general and is not limited to specific entanglement measures or to specific bipartite spaces but applies to all the bipartite Hilbert spaces and extends to the sets of fully separable states in multipartite spaces. Moreover, the result makes it possible to define natural measures that vanish exactly on the separable states.

It may be possible to extend our results into higher-rank mixed states: For example, it is possible to look at portions of the higher-rank states (e.g., a nondegenerate rank 3 state defines three Bloch spheres, each corresponding to two out of the three eigenstates); and it is possible to analyze higher-rank states that are $\epsilon$ close ($\epsilon \ll 1$) to rank 2 states.

Our analysis identifies the set of Bloch-sphere neighbor states of any rank 2 state (namely, the set of states in its Bloch sphere). Such Bloch-sphere neighbor states may be useful for various protocols: For example, entanglement purification or error correction protocols may first turn the state into a Bloch-sphere neighbor state of desired properties (e.g., more entangled) and then operate on that Bloch-sphere neighbor state. Those possibilities may be explored by future research.

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