Geometry of Entanglement in the Bloch Sphere

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Joint work with:
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We look at the set of separable states in a Bloch sphere.

For example:

\[ |00\rangle \quad |\Phi_-\rangle \quad |\Phi_+\rangle \quad |11\rangle \]
Topics

- Introduction
- Generalized Bloch sphere
- Our work: general result for bipartite Hilbert spaces
- Extension to multipartite Hilbert spaces
- Previous works
- Conclusion
Entanglement is important for:

- The foundations of quantum mechanics:
  - The EPR paradox
  - Bell’s inequality

- Quantum communication:
  - Quantum teleportation
  - Quantum cryptography

- Quantum computers
Let $\rho$ be a mixed bipartite state. That is, a mixture of states in the bipartite Hilbert space $H_A \otimes H_B$.

$\rho$ is said to be **separable** if:

$$\rho = \sum_j p_j |\psi_j>_A <\psi_j|_A \otimes |\psi_j>_B <\psi_j|_B$$

Otherwise, $\rho$ is said to be **entangled**.
Bloch Sphere for One Qubit

- **Bloch sphere** (or Poincaré sphere) is a geometrical representation of $H_2 = \text{Span}\{|0\>, |1\>\}$ and of its mixtures.

The point $(x, y, z)$ corresponds to the state:

$$
\frac{1}{\sqrt{3}} \begin{bmatrix}
1 + z & x - iy \\
1 - z & x + iy
\end{bmatrix}
$$
Properties of the Bloch Sphere

- A **diameter** connects **orthogonal** states.
- A **convex combination** of two states lies **on the line** connecting between them.
We look at a rank-2 state $\rho$.

Writing its spectral decomposition, we get:

\[ \rho = q \cdot |\psi_0><\psi_0| + (1-q) \cdot |\psi_1><\psi_1| \]

\{\psi_0, \psi_1\} is an orthonormal set.

This state lies inside a unique Bloch sphere:

- The sphere represents $\text{Span}\{|\psi_0>, |\psi_1>\}$.
- Inside the sphere are the mixtures of those states.
Three equivalent notions:

\[ \rho = q |\psi_0\rangle\langle\psi_0| + (1 - q) |\psi_1\rangle\langle\psi_1| \]

Rank-2 mixed state

Span \{ |\psi_0\rangle, |\psi_1\rangle \}

Two-dimensional Hilbert space

Generalized Bloch sphere
Our results apply to any bipartite system, represented by a Hilbert space $H_A \otimes H_B$:

For each rank-2 state, there are **five possibilities** for the Bloch sphere:

1. No entanglement
2. One separable line – diameter
3. One separable line – non-diameter
4. One separable point
5. Completely entangled subspace
Let $A$ and $B$ be two subsystems.

Let $\rho$ be a state in the system $H_A \otimes H_B$.

Let $\rho^T_B$ be the partial transpose of $\rho$ with respect to the subsystem $B$.

If $\rho^T_B$ is not a positive operator,

then $\rho$ must be entangled.
In $H_2 \otimes H_2$, $H_2 \otimes H_3$ and $H_3 \otimes H_2$:

- If $\rho$ is entangled, then $\rho^{TB}$ is not a positive operator.

However, in other Hilbert spaces, there exist entangled states with a positive partial transpose.

Those states are bound entangled: their entanglement cannot be distilled.
Lemma:

Let $\rho$ be a mixture of states in $H_A \otimes H_B$.

If there are $|\phi_A>, |\psi_A> \in H_A$ and $|\phi_B>, |\psi_B> \in H_B$ such that:

- $<\phi_A \phi_B | \rho | \phi_A \phi_B> = 0$, and
- $<\phi_A \psi_B | \rho | \psi_A \phi_B> \neq 0$,

then

- $\rho$ is entangled.
A Weaker Entanglement Criterion

Proof:

We define $|\phi_B^\rangle$ and $|\psi_B^\rangle$, as follows:

if, in the computational basis:

- $|\phi_B^\rangle = \sum_j a_j |j_B^\rangle$
- $|\psi_B^\rangle = \sum_j b_j |j_B^\rangle$

Then:

- $|\phi_B^{*^\rangle} = \sum_j a_j^{*} |j_B^\rangle$
- $|\psi_B^{*^\rangle} = \sum_j b_j^{*} |j_B^\rangle$
Proof – continued:

- Let: \( C_{ijkl} = |i><j| \otimes |k><l| \)
- Then: \( C_{ijkl}^{TB} = |i><j| \otimes |l><k| \)
- We calculate:
  - \( <\phi_A \phi_B^* | C_{ijkl}^{TB} | \psi_A \psi_B^*> \)
  - \( = <\phi_A|i><j|\psi_A> \cdot <\phi_B^*|l><k|\psi_B^*> \)
  - \( = <\phi_A|i><j|\psi_A> \cdot <\psi_B|k><l|\phi_B> \)
  - \( = <\phi_A \psi_B | C_{ijkl} | \psi_A \phi_B> \)
Proof – continued:

- By linearity:
  - $\langle \phi_A \phi_B^* | \rho^T_B | \phi_A \phi_B^* \rangle = \langle \phi_A \phi_B | \rho | \phi_A \phi_B \rangle = 0$
  - $\langle \phi_A \phi_B^* | \rho^T_B | \psi_A \psi_B^* \rangle = \langle \phi_A \psi_B | \rho | \psi_A \phi_B \rangle \neq 0$

- Therefore, $\rho^T_B$ cannot be positive semidefinite!

- By the Peres-Horodecki criterion, $\rho$ must be entangled.
Let $\rho$ be a rank-2 mixed state.

All the pure states appearing in any decomposition of $\rho$ lie on the same Bloch sphere.

This proves that each rank-2 mixed state is included in exactly one Bloch sphere.
Another Property of Bloch Spheres

In particular:
- Let $\rho$ be a rank-2 mixed state.
- If $\rho$ is separable,
- then there exist two different separable (pure) states on the Bloch sphere.
We analyze the **set of separable states**. There are **five classes** of Bloch spheres:

- **Class 1 – no entanglement:**

![Bloch Sphere Diagram]

|00⟩ |0−⟩ |0+⟩ |01⟩
Bloch Spheres – Classification

Class 2 – one line of separability, connecting two orthogonal states:

\[ |00\rangle \]
\[ |\Phi_-angle \]
\[ |11\rangle \]
\[ |\Phi_+angle \]
Class 3 – one line of separability, connecting two non-orthogonal states:
**Class 4** – one separable point, that must be a pure state:

\[ |00\rangle \]

\[ \alpha |01\rangle + \beta |10\rangle \]

\[ \alpha \beta \neq 0 \]
Bloch Spheres – Classification

- **Class 5** – completely entangled subspace:

\[
\frac{|00\rangle + |11\rangle}{\sqrt{2}}
\]

\[
\frac{|02\rangle + |10\rangle}{\sqrt{2}}
\]

(does not exist for two qubits)
Our Central Theorem

Theorem:

- Any rank-2 bipartite state $\rho_0$ (a mixture of states in $H_A \otimes H_B$) belongs to one of those five classes.
- Namely, no other classes exist.
Our Central Theorem

Proof:

- **Case 1**: no separable mixed state exists inside the Bloch sphere of $\rho_0$.
- Therefore, there cannot be two (or more) separable pure states!
  - Otherwise, the line connecting them is separable.
- Two possibilities remain:
  - One separable pure state (Class 4)
  - No separable states at all (Class 5)
Proof – continued:

- **Case 2**: a separable **mixed** state $\rho_1$ exists inside the Bloch sphere of $\rho_0$.
- Therefore, there are two **different** separable **pure** states:
  - $|\psi\rangle = |\psi_A\rangle |\psi_B\rangle$
  - $|\phi\rangle = |\phi_A\rangle |\phi_B\rangle$
Our Central Theorem

Proof – continued:

- All the mixed states in the Bloch sphere:
  - \( \rho = a_{00} |\psi><\psi| + a_{01} |\psi><\phi| + a_{10} |\phi><\psi| + a_{11} |\phi><\phi| \)

- Two possibilities exist:
  - \( |\psi_A> \cong |\phi_A> \) or \( |\psi_B> \cong |\phi_B> \) (equality up to global phase):
    corresponds to Class 1
  - \( |\psi_A> \not\cong |\phi_A> \) and \( |\psi_B> \not\cong |\phi_B> \) :
    corresponds to Class 2 or Class 3 (proved below)
Our Central Theorem

Proof – continued:

• Assume that $|\psi_A\rangle \not\equiv |\phi_A\rangle$ and $|\psi_B\rangle \not\equiv |\phi_B\rangle$.

• Define:
  • $|\phi_A'\rangle$ such that $\langle \phi_A | \phi_A' \rangle = 0$ and $\langle \psi_A | \phi_A' \rangle \neq 0$
  • $|\phi_B'\rangle$ such that $\langle \phi_B | \phi_B' \rangle = 0$ and $\langle \psi_B | \phi_B' \rangle \neq 0$
  • Similarly, $|\psi_A'\rangle$ and $|\psi_B'\rangle$

• Let $\rho$ be a state inside the sphere:
  • $\rho = a_{00} |\psi\rangle\langle\psi| + a_{01} |\psi\rangle\langle\phi| + a_{10} |\phi\rangle\langle\psi| + a_{11} |\phi\rangle\langle\phi|$
Our Central Theorem

Proof – continued:

- Let $\rho$ be a state inside the sphere:
  
  $\rho = a_{00} |\psi><\psi| + a_{01} |\psi><\phi| + a_{10} |\phi><\psi| + a_{11} |\phi><\phi|$

- If $a_{01} = a_{10} = 0$ (the line connecting $|\psi>$ and $|\phi>$), $\rho$ is **separable**.
Our Central Theorem

Proof – continued:

Let \( \rho \) be a state inside the sphere:

\[
\rho = a_{00} |\psi><\psi| + a_{01} |\psi><\phi| + a_{10} |\phi><\psi| + a_{11} |\phi><\phi|
\]

If \( a_{10} \neq 0 \) (all the states outside that line), then:

\[
<\psi_A' \phi_B' | \rho | \psi_A' \phi_B'> = 0
\]

\[
<\psi_A' \psi_B' | \rho | \phi_A' \phi_B'> = a_{10} <\psi_A' \psi_B'|\phi><\psi|\phi_A' \phi_B' > \neq 0
\]

By our Lemma, \( \rho \) is entangled.

Reminder – the Lemma:

If there are \( |\phi_A>, |\psi_A> \in H_A \) and \( |\phi_B>, |\psi_B> \in H_B \) such that:

\[
<\phi_A \phi_B | \rho | \phi_A \phi_B > = 0, \text{ and } <\phi_A \psi_B | \rho | \psi_A \phi_B > \neq 0,
\]

then \( \rho \) is entangled.
Let $\rho$ be a mixed multipartite state. That is, a mixture of states in the multipartite Hilbert space $H_{A_1} \otimes H_{A_2} \otimes \ldots \otimes H_{A_m}$.

$\rho$ is said to be **fully separable** if:

$$\rho = \sum_j p_j |\psi_j\rangle_{A_1} \langle \psi_j|_{A_1} \otimes |\psi_j\rangle_{A_2} \langle \psi_j|_{A_2} \otimes \ldots \otimes |\psi_j\rangle_{A_m} \langle \psi_j|_{A_m}$$

Otherwise, $\rho$ is said to be **entangled**.
Multipartite Entanglement

Examples:

- The tripartite states $|000\rangle$, $|011\rangle$, and $|111\rangle$ are fully separable.
- The tripartite states $[|000\rangle \pm |111\rangle] / \sqrt{2}$ are entangled with respect to all the bipartite partitions: $\{\{1, 2\}, \{3\}\}$, $\{\{1, 3\}, \{2\}\}$, and $\{\{1\}, \{2, 3\}\}$.
- The tripartite states $|\Phi_\pm\rangle = [|000\rangle \pm |011\rangle] / \sqrt{2}$ are entangled with respect to the bipartite partitions $\{\{1, 2\}, \{3\}\}$ and $\{\{1, 3\}, \{2\}\}$, and separable with respect to the partition $\{\{1\}, \{2, 3\}\}$.
Our Theorem discusses bipartite entanglement. However, it can be easily extended to multipartite entanglement:

- The set of fully separable states belongs to one of the five classes.
The line connecting $|000\rangle$ and $|111\rangle$ is fully separable.

All the other states are entangled with respect to any bipartite partition.
Multipartite: Example 2

- The line connecting $|000\rangle$ and $|011\rangle$ is **fully separable**.
- All the other states are:
  - **entangled** with respect to the partitions $\{\{1, 2\}, \{3\}\}$ and $\{\{1, 3\}, \{2\}\}$; and
  - **separable** with respect to the partition $\{\{1\}, \{2, 3\}\}$. 
Osterloh, Siewert and Uhlmann analyzed entanglement of rank-2 mixed states:
- Based on entanglement measures that are polynomials in the coefficients of the state.
- Example: the concurrence (for two qubits):
  \[ C(\psi) = 2 |a_{00} a_{11} - a_{01} a_{10}| \]

Regula and Adesso further analyzed the entanglement measures in Bloch sphere.
- Our work applies to any bipartite space.
Osterloh, Siewert and Uhlmann:

Regula and Adesso:
**Conclusion**

- **All** the Bloch spheres in **all** the bipartite Hilbert spaces belong to **five** classes:

```
|00⟩  |00⟩  |00⟩
|01⟩  |Φ−⟩  |Φ+⟩
|0−⟩  |0+⟩  |++⟩
```

- \( |0⟩ + β|1⟩ \neq 0 \)

- \( \frac{|00⟩ + |11⟩}{\sqrt{2}} \)

- \( \frac{|01⟩ + |10⟩ + |11⟩}{\sqrt{3}} \)

- \( \frac{|0⟩ + |1⟩}{\sqrt{2}} \)
Thank you!