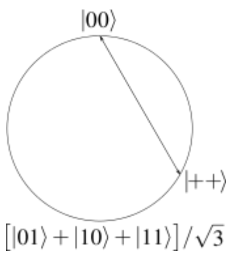
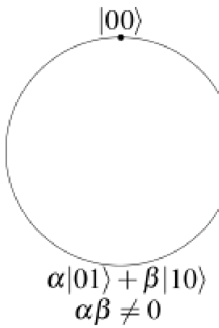
Geometry of Entanglement in the Bloch Sphere



Rotem Liss

22 May, 2019

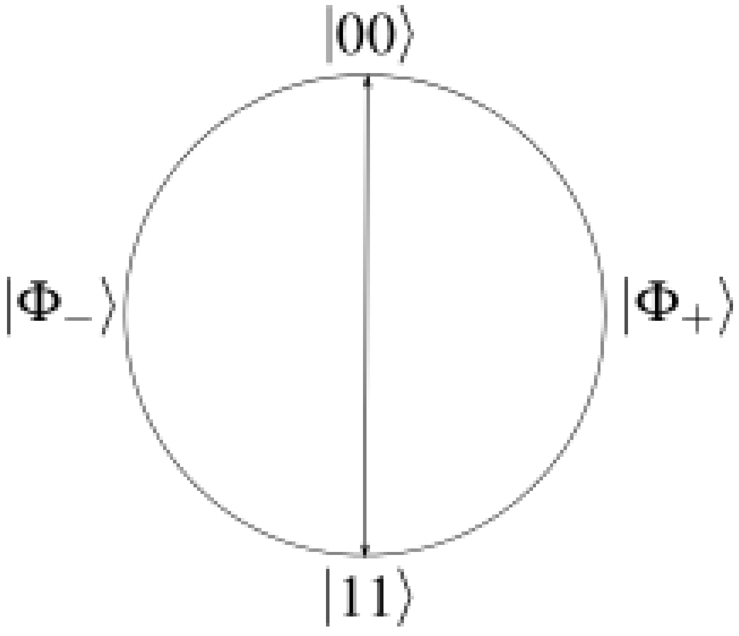


Joint work with:

Michel Boyer and Tal Mor

Preface: Bloch-sphere Entanglement

- We look at **the set of separable states** in a Bloch sphere.
- For example:



Topics

- Introduction
- Generalized Bloch sphere
- Our work: general result for bipartite Hilbert spaces
- Extension to multipartite Hilbert spaces
- Previous works
- Conclusion

Introduction

Entanglement is important for:

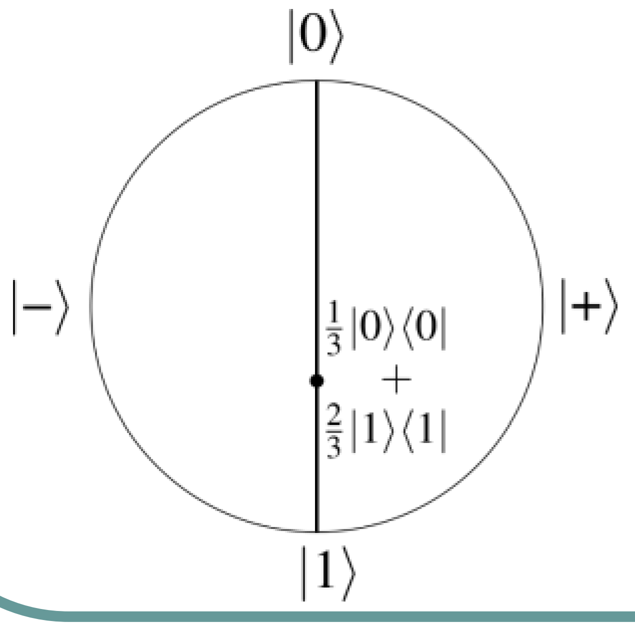
- The <u>foundations</u> of quantum mechanics:
  - The EPR paradox
  - Bell's inequality
- Quantum <u>communication</u>:
  - Quantum teleportation
  - Quantum cryptography
- Quantum <u>computers</u>

Introduction: Bipartite Entanglement

- Let ρ be a mixed bipartite state.
  - That is, a mixture of states in the bipartite Hilbert space $H_A \otimes H_B$.
- ρ is said to be **separable** if:
  - $\rho = \sum_j p_j |\psi_j\rangle_A\langle\psi_j|_A \otimes |\psi_j\rangle_B\langle\psi_j|_B$
- Otherwise, ρ is said to be **entangled**.

Bloch Sphere for One Qubit

- **Bloch sphere** (or Poincaré sphere) is a geometrical representation of H_2 = Span{|0>, |1>} and of its mixtures.

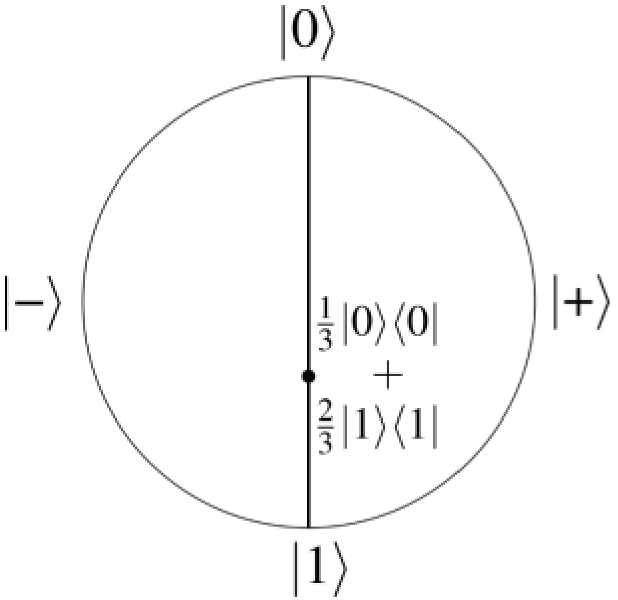


The point (x, y, z) corresponds to the state:

$$\frac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}$$

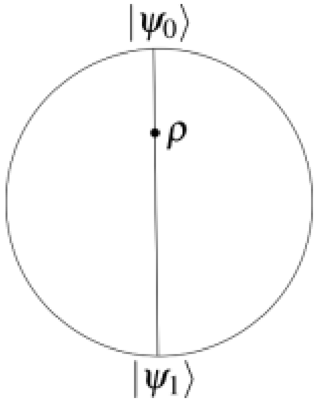
Properties of the Bloch Sphere

- A <u>diameter</u> connects **orthogonal** states.
- A <u>convex combination</u> of two states lies **on the line** connecting between them.



Bloch Sphere for a Rank-2 State

- We look at a rank-2 state ρ.
- Writing its spectral decomposition, we get:
  - $\rho = q \cdot |\psi_0\rangle\langle\psi_0| + (1-q) \cdot |\psi_1\rangle\langle\psi_1|$
  - $\{|\psi_0\rangle, |\psi_1\rangle\}$ is an orthonormal set.
- This state lies inside a **unique** Bloch sphere:
  - The sphere represents **Span$\{|\psi_0\rangle, |\psi_1\rangle\}$**.
  - Inside the sphere are the **mixtures** of those states.

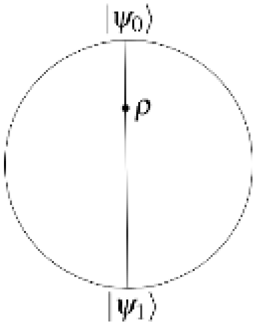


Bloch Sphere for a Rank-2 State

Three **equivalent** notions:

$$\rho = q\, |\psi_0\rangle\langle\psi_0| + (1 - q)\, |\psi_1\rangle\langle\psi_1|$$

Rank-2 mixed state



Span $\{|\psi_0\rangle, |\psi_1\rangle\}$

Two-dimensional Hilbert space

Generalized Bloch sphere

Our Work: General Result

- Our results apply to **any bipartite** system, represented by a Hilbert space $H_A \otimes H_B$:
- For each rank-2 state, there are **five possibilities** for the Bloch sphere:
  1. No entanglement
  2. One separable line – diameter
  3. One separable line – non-diameter
  4. One separable point
  5. Completely entangled subspace

Peres-Horodecki Criterion

- Let A and B be two subsystems.
- Let ρ be a state in the system $H_A \otimes H_B$.
- Let ρ^{T_B} be the partial transpose of ρ with respect to the subsystem B.
- **If**
  - ρ^{T_B} is not a positive operator,
- **then**
  - ρ must be entangled.

Peres-Horodecki Criterion

- In $H_2 \otimes H_2$, $H_2 \otimes H_3$ and $H_3 \otimes H_2$:
  - **If** ρ is entangled, **then** ρ^{T_B} is not a positive operator.
- **However**, in other Hilbert spaces, there exist entangled states with a positive partial transpose.
  - Those states are **bound entangled**: their entanglement cannot be distilled.

A Weaker Entanglement Criterion

Lemma:

- Let ρ be a mixture of states in $H_A \otimes H_B$.
- **If** there are $|\phi_A\rangle, |\psi_A\rangle \in H_A$ and
  $|\phi_B\rangle, |\psi_B\rangle \in H_B$ such that:
  - $\langle\phi_A \phi_B | \rho | \phi_A \phi_B\rangle = 0$, **and**
  - $\langle\phi_A \psi_B | \rho | \psi_A \phi_B\rangle \neq 0$,
- **then**
  - ρ is entangled.

A Weaker Entanglement Criterion

Proof:

- We define $|\phi_B{}^*\rangle$ and $|\psi_B{}^*\rangle$, as follows: if, in the computational basis:
  - $|\phi_B\rangle = \sum_j a_j \, |j_B\rangle$
  - $|\psi_B\rangle = \sum_j b_j \, |j_B\rangle$
- Then:
  - $|\phi_B{}^*\rangle = \sum_j a_j{}^* \, |j_B\rangle$
  - $|\psi_B{}^*\rangle = \sum_j b_j{}^* \, |j_B\rangle$

A Weaker Entanglement Criterion

Proof – continued:

- Let: $C_{ijkl} = |i\rangle\langle j| \otimes |k\rangle\langle l|$
- Then: $C_{ijkl}^{T_B} = |i\rangle\langle j| \otimes |l\rangle\langle k|$
- We calculate:
  - $\langle \phi_A\, \phi_B^* \,|\, C_{ijkl}^{T_B} \,|\, \psi_A\, \psi_B^* \rangle$
  - $= \langle \phi_A|i\rangle\langle j|\psi_A\rangle \cdot \langle \phi_B^*|l\rangle\langle k|\psi_B^*\rangle$
  - $= \langle \phi_A|i\rangle\langle j|\psi_A\rangle \cdot \langle \psi_B|k\rangle\langle l|\phi_B\rangle$
  - $= \langle \phi_A\, \psi_B \,|\, C_{ijkl} \,|\, \psi_A\, \phi_B \rangle$

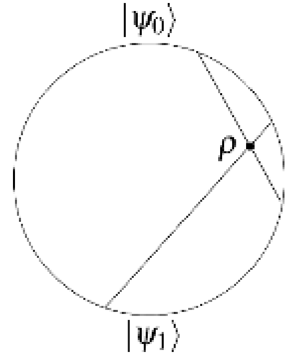
A Weaker Entanglement Criterion

Proof – continued:

- By linearity:
  - $\langle \phi_A\ \phi_B^* \mid \rho^{T_B} \mid \phi_A\ \phi_B^* \rangle = \langle \phi_A\ \phi_B \mid \rho \mid \phi_A\ \phi_B \rangle = 0$
  - $\langle \phi_A\ \phi_B^* \mid \rho^{T_B} \mid \psi_A\ \psi_B^* \rangle = \langle \phi_A\ \psi_B \mid \rho \mid \psi_A\ \phi_B \rangle \neq 0$
- Therefore, ρ^{T_B} cannot be positive semidefinite!
- By the Peres-Horodecki criterion, ρ must be **entangled**.

Another Property of Bloch Spheres

- Let ρ be a rank-2 **mixed state**.
- All the pure states appearing in any decomposition of ρ lie **on the same Bloch sphere**.
- This proves that each rank-2 mixed state is included in **exactly one** Bloch sphere.



Another Property of Bloch Spheres

In particular:

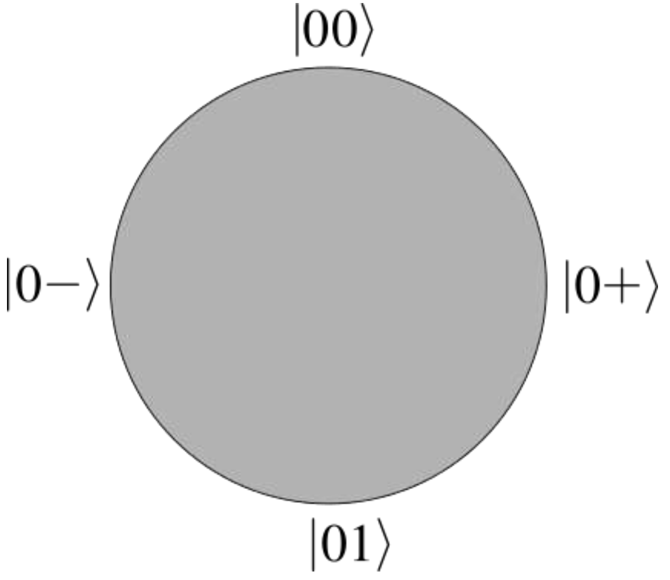
- Let ρ be a rank-2 mixed state.
- **If**
  - ρ is separable,
- **then**
  - there exist **two <u>different</u> separable (<u>pure</u>) states** on the Bloch sphere.

Bloch Spheres – Classification

We analyze the **set of separable states**.

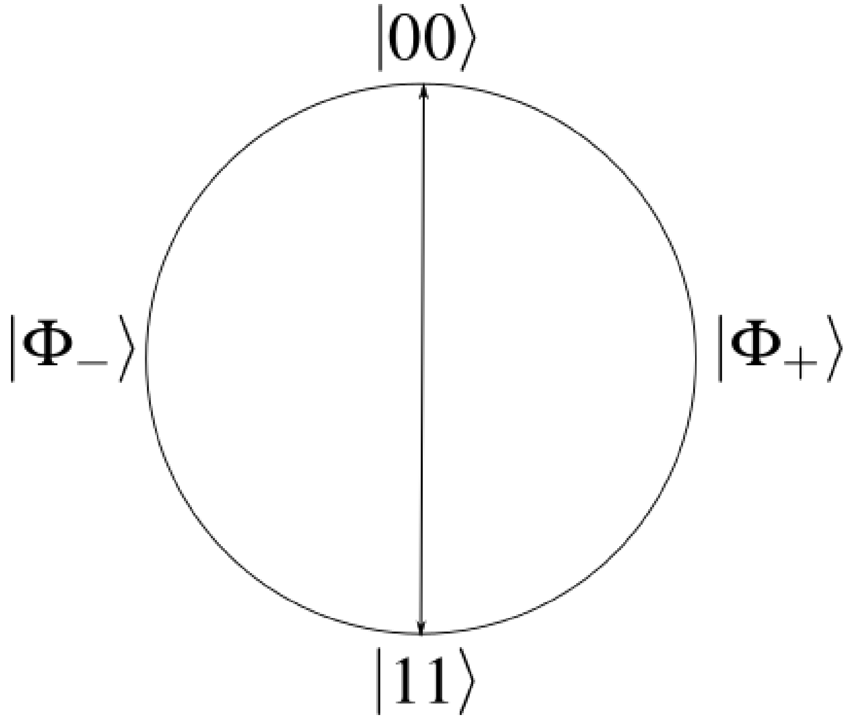
There are five classes of Bloch spheres:

- Class 1 – **no entanglement**:



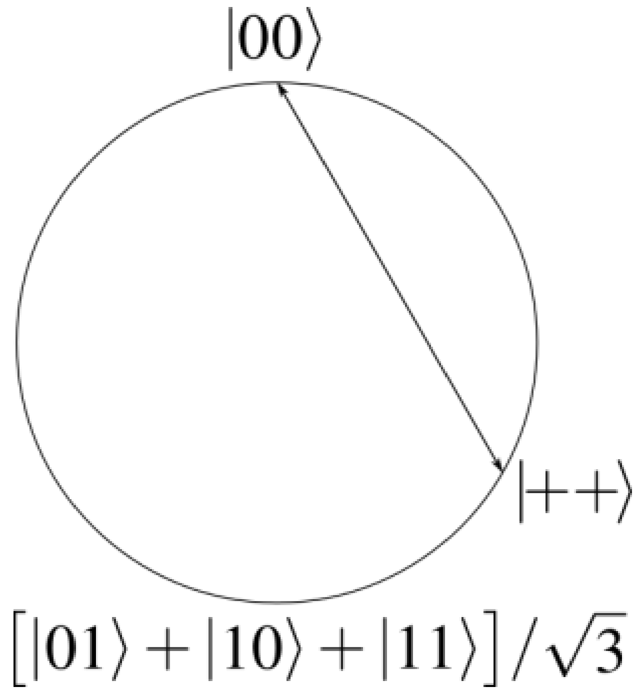
Bloch Spheres – Classification

- Class 2 – **one line of separability**, connecting two orthogonal states:



Bloch Spheres – Classification

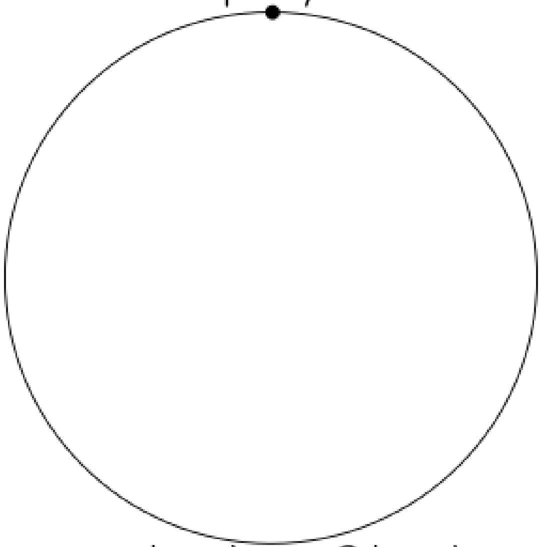
- <u>Class 3</u> – **one line of separability**, connecting <u>two non-orthogonal states</u>:



Bloch Spheres – Classification

- Class 4 – **one separable point**, that must be a pure state:

$|00\rangle$



$\alpha|01\rangle + \beta|10\rangle$

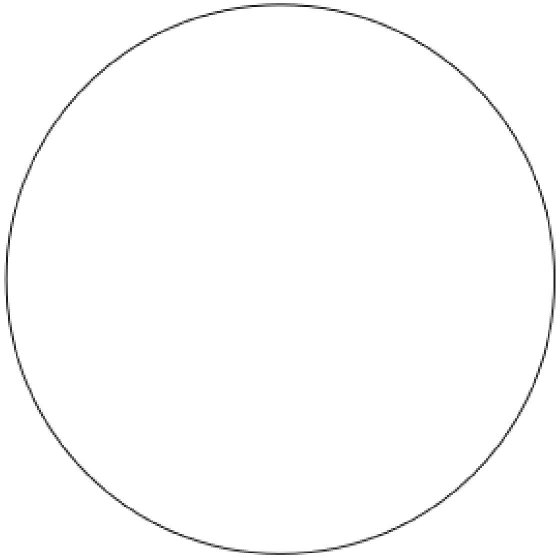
$\alpha\beta \neq 0$

Bloch Spheres – Classification

- Class 5 – **completely entangled subspace**:

(does not exist for two qubits)

$$\left[|00\rangle + |11\rangle\right]/\sqrt{2}$$



$$\left[|02\rangle + |10\rangle\right]/\sqrt{2}$$

Our Central Theorem

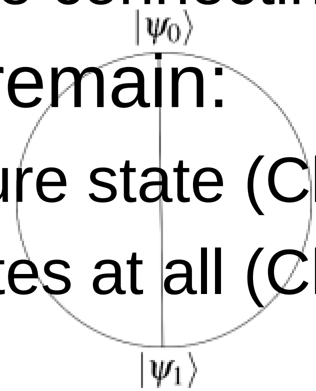
Theorem:

- Any rank-2 bipartite state ρ_0 (a mixture of states in $H_A \otimes H_B$) belongs to **one of those five classes**.
- Namely, no other classes exist.

Our Central Theorem

Proof:

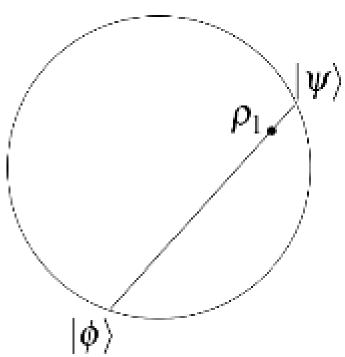
- Case 1: no separable **mixed** state exists inside the Bloch sphere of ρ_0.
- Therefore, there cannot be two (or more) separable **pure** states!
  - Otherwise, the line connecting them is separable.
- Two possibilities remain:
  - One separable pure state (Class 4)
  - No separable states at all (Class 5)



Our Central Theorem

Proof – continued:

- Case 2: a separable **mixed** state ρ_1 exists inside the Bloch sphere of ρ_0.
- Therefore, there are two different separable **pure** states:
  - $|\psi\rangle = |\psi_A\rangle |\psi_B\rangle$
  - $|\phi\rangle = |\phi_A\rangle |\phi_B\rangle$



Our Central Theorem

Proof – continued:

- All the mixed states in the Bloch sphere:
  - $\rho = a_{00}\,|\psi\rangle\langle\psi| + a_{01}\,|\psi\rangle\langle\phi| + a_{10}\,|\phi\rangle\langle\psi| + a_{11}\,|\phi\rangle\langle\phi|$
- Two possibilities exist:
  - $|\psi_A\rangle \cong |\phi_A\rangle$ <u>or</u> $|\psi_B\rangle \cong |\phi_B\rangle$
(equality up to global phase):
corresponds to **Class 1**
  - $|\psi_A\rangle \ncong |\phi_A\rangle$ <u>and</u> $|\psi_B\rangle \ncong |\phi_B\rangle$:
corresponds to **Class 2** or **Class 3** (*proved below*)

Our Central Theorem

Proof – continued:

- Assume that $|\psi_A\rangle \not\approx |\phi_A\rangle$ <u>and</u> $|\psi_B\rangle \not\approx |\phi_B\rangle$.
- Define:
  - $|\phi_A'\rangle$ such that $\langle\phi_A|\phi_A'\rangle = 0$ and $\langle\psi_A|\phi_A'\rangle \neq 0$
  - $|\phi_B'\rangle$ such that $\langle\phi_B|\phi_B'\rangle = 0$ and $\langle\psi_B|\phi_B'\rangle \neq 0$
  - Similarly, $|\psi_A'\rangle$ and $|\psi_B'\rangle$
- Let ρ be a state inside the sphere:
  - $\rho = a_{00} |\psi\rangle\langle\psi| + a_{01} |\psi\rangle\langle\phi| + a_{10} |\phi\rangle\langle\psi| + a_{11} |\phi\rangle\langle\phi|$

Our Central Theorem

Proof – continued:

- Let ρ be a state inside the sphere:
  - $\rho = a_{00}\,|\psi\rangle\langle\psi| + a_{01}\,|\psi\rangle\langle\phi| + a_{10}\,|\phi\rangle\langle\psi| + a_{11}\,|\phi\rangle\langle\phi|$
- If $a_{01} = a_{10} = 0$ (the line connecting $|\psi\rangle$ and $|\phi\rangle$), ρ is **separable**.

Our Central Theorem

Proof – continued:

- Let ρ be a state inside the sphere:
  - $\rho = a_{00}\,|\psi\rangle\langle\psi| + a_{01}\,|\psi\rangle\langle\phi| + a_{10}\,|\phi\rangle\langle\psi| + a_{11}\,|\phi\rangle\langle\phi|$
- If $a_{10} \neq 0$ (all the states outside that line), then:
  - $\boldsymbol{\langle\psi_A'\,\phi_B'\,|\,\rho\,|\,\psi_A'\,\phi_B'\rangle = 0}$
  - $\boldsymbol{\langle\psi_A'\,\psi_B'\,|\,\rho\,|\,\phi_A'\,\phi_B'\rangle} = a_{10}\,\langle\psi_A'\,\psi_B'|\phi\rangle\langle\psi|\phi_A'\,\phi_B'\rangle \boldsymbol{\neq 0}$
  - By our Lemma, ρ is **entangled**. □

Reminder – the Lemma:

- **If** there are $|\phi_A\rangle, |\psi_A\rangle \in H_A$ and $|\phi_B\rangle, |\psi_B\rangle \in H_B$ such that: $\langle\phi_A\,\phi_B\,|\,\rho\,|\,\phi_A\,\phi_B\rangle = 0$, **and** $\langle\phi_A\,\psi_B\,|\,\rho\,|\,\psi_A\,\phi_B\rangle \neq 0$,
- **then** ρ is entangled.

Multipartite Entanglement

- Let ρ be a mixed multipartite state.
  - That is, a mixture of states in the multipartite Hilbert space $H_{A_1} \otimes H_{A_2} \otimes \ldots \otimes H_{A_m}$.
- ρ is said to be **fully separable** if:
  - $\rho = \sum_j p_j |\psi_j>_{A_1}<\psi_j|_{A_1} \otimes |\psi_j>_{A_2}<\psi_j|_{A_2} \otimes \ldots \otimes |\psi_j>_{A_m}<\psi_j|_{A_m}$
- Otherwise, ρ is said to be **entangled**.

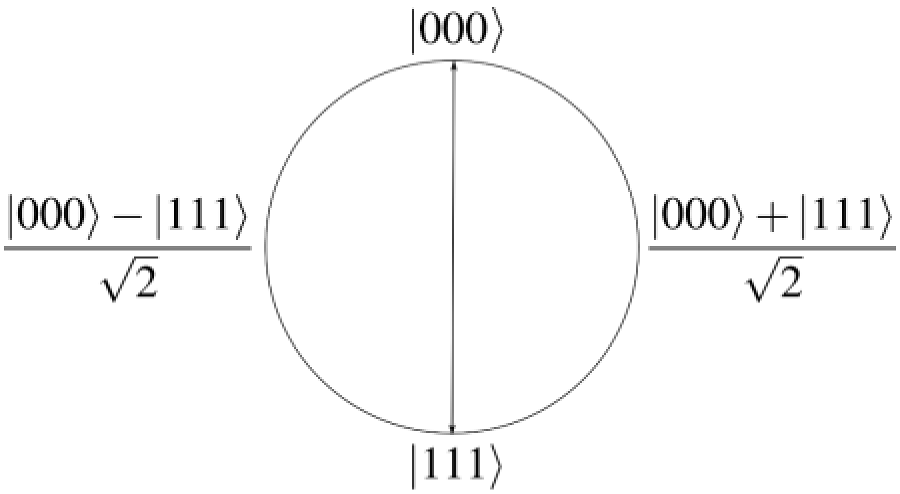
Multipartite Entanglement

- Examples:
  - The tripartite states **|000>**, **|011>**, and **|111>** are **fully separable**.
  - The tripartite states **[|000> ± |111>] / √2** are **entangled** with respect to all the <u>bipartite partitions</u>: {{1, 2}, {3}}, {{1, 3}, {2}}, and {{1}, {2, 3}}.
  - The tripartite states **$|0\Phi_{\pm}\rangle$ = [|000> ± |011>] / √2** are **entangled** with respect to the <u>bipartite partitions</u> {{1, 2}, {3}} and {{1, 3}, {2}}, and <u>separable</u> with respect to the partition {{1}, {2, 3}}.

Multipartite Entanglement

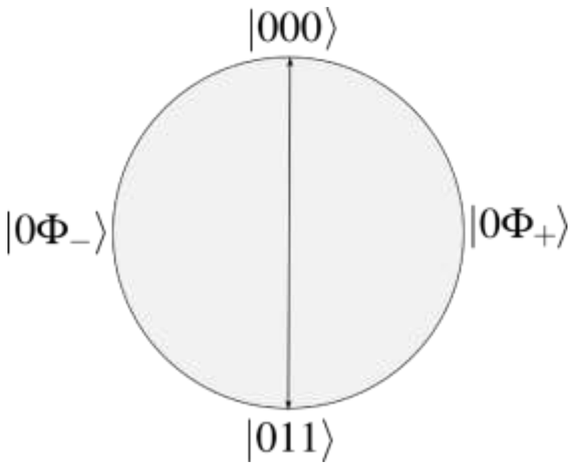
- Our Theorem discusses bipartite entanglement.
- However, it can be **easily extended** to multipartite entanglement:
  - The set of fully separable states belongs to one of the five classes.

Multipartite: Example 1



- The line connecting |000> and |111> is **fully separable**.
- All the other states are **entangled** with respect to <u>any bipartite partition</u>.

Multipartite: Example 2



- The line connecting |000> and |011> is **fully separable**.
- All the other states are:
  - **entangled** with respect to the partitions {{1, 2}, {3}} and {{1, 3}, {2}}; and
  - **separable** with respect to the partition {{1}, {2, 3}}.

Previous Works

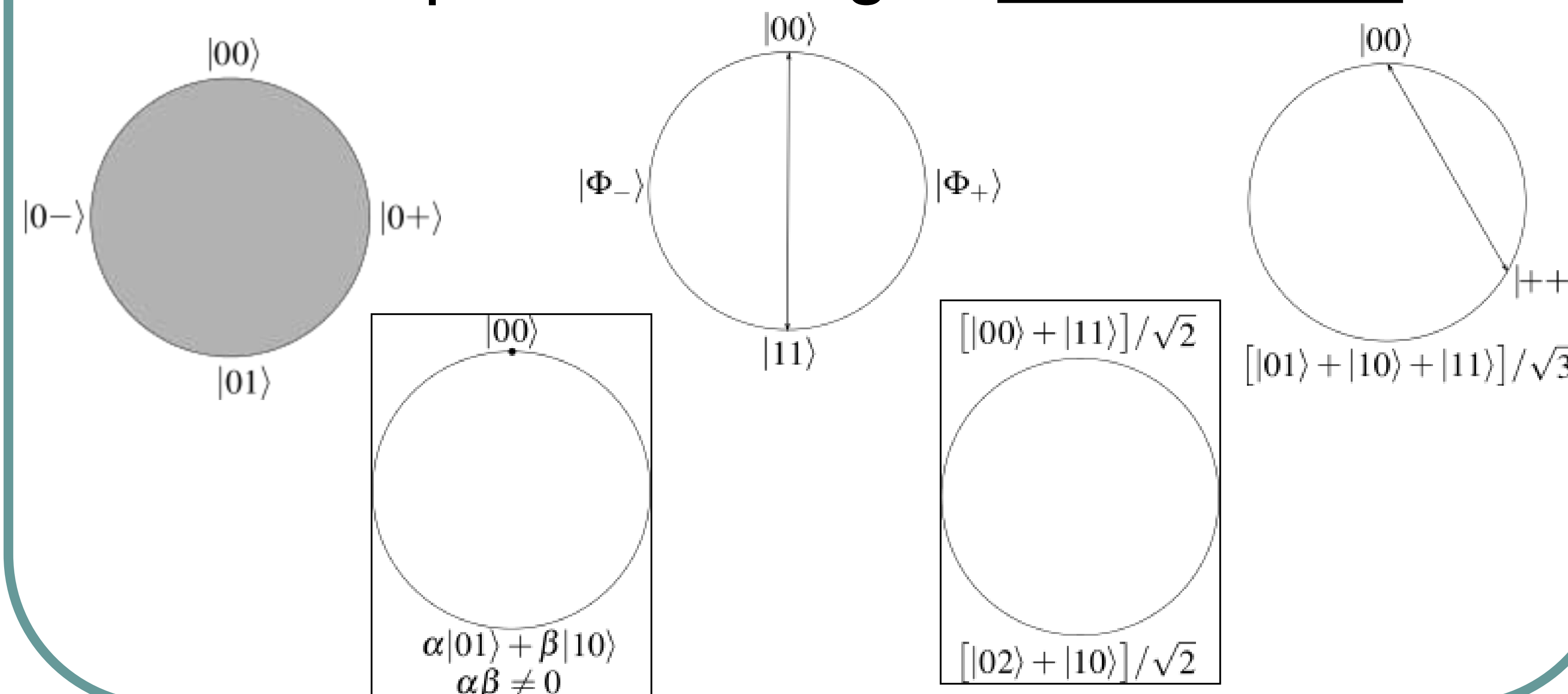
- Osterloh, Siewert and Uhlmann analyzed entanglement of rank-2 mixed states:
  - Based on entanglement measures that are polynomials in the coefficients of the state.
  - Example: the concurrence (for two qubits): $C(\psi) = 2\,|a_{00}\, a_{11} - a_{01}\, a_{10}|$
- Regula and Adesso further analyzed the entanglement measures in Bloch sphere.
- Our work applies to **any** bipartite space.

Previous Works – References

- Osterloh, Siewert and Uhlmann:
  - Lohmayer, R., Osterloh, A., Siewert, J., & Uhlmann, A. “Entangled three-qubit states without concurrence and three-tangle.” *Physical review letters*, *97*(26), 260502 (2006).
  - Osterloh, A., Siewert, J., & Uhlmann, A. “Tangles of superpositions and the convex-roof extension.” *Physical Review A*, *77*(3), 032310 (2008).
- Regula and Adesso:
  - Regula, B., & Adesso, G. “Entanglement quantification made easy: Polynomial measures invariant under convex decomposition.” *Physical review letters*, *116*(7), 070504 (2016).

Conclusion

- **All** the Bloch spheres in **all** the bipartite Hilbert spaces belong to <u>five classes</u>:



Thank you!